# DIGITAL ARITHMETIC

LECTURE 06-2

JIM FIX, REED COLLEGE CS2-F20

# 2:1 MULTIPLEXER CIRCUIT

RECALL...

Let's express the logic using AND, OR, NOT...

$$o := \left\{ \begin{array}{c} \bar{s} \cdot i_0 \cdot \bar{i}_1 + \bar{s} \cdot i_0 \cdot i_1 \\ + \\ s \cdot \bar{i}_0 \cdot i_1 + s \cdot i_0 \cdot i_1 \end{array} \right\} = \begin{array}{c} \boxed{\bar{s} \cdot i_0} \\ + \\ s \cdot i_1 \end{array}$$

Note that:

$$\bar{s} \cdot i_0 \cdot \bar{i}_1 + \bar{s} \cdot i_0 \cdot i_1 = \bar{s} \cdot i_0 \left( \bar{i}_1 + i_1 \right) = \bar{s} \cdot i_0 \cdot 1 = \bar{s} \cdot i_0$$

## Properties of boolean algebra:

| | | |
|---|---|---|
| $a \cdot b = b \cdot a$ | commutative | $a + b = b + a$ |
| $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ | associative | $(a+b)+c = a + (b+c)$ |
| $1 \cdot a = a$ | identity | $0 + a = a$ |
| $0 \cdot a = 0$ | anihilator | $1 + a = 1$ |
| $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ | distributive | $a + (b \cdot c) = (a+b) \cdot (a+c)$ |
| $a \cdot a = a$ | idempotent | $a + a = a$ |
| $a \cdot \bar{a} = 0$ | complement | $a + \bar{a} = 1$ |
| $\overline{a \cdot b} = \bar{a} + \bar{b}$ | DeMorgan's | $\overline{(a+b)} = \bar{a} \cdot \bar{b}$ |

**others:**

$\bar{\bar{a}} = a$

$a \cdot b + a \cdot \bar{b} = a$

$\bar{0} = 1$

$a + ab = a$

Proof that $a \cdot (b+c) = a \cdot b + a \cdot c$

| $a$ | $b$ | $c$ | $b+c$ | $a \cdot (b+c)$ | $a \cdot b$ | $a \cdot c$ | $a \cdot b + a \cdot c$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | | | | | |
| 0 | 0 | 1 | | | | | |
| 0 | 1 | 0 | | | | | |
| 0 | 1 | 1 | | | | | |
| 1 | 0 | 0 | | | | | |
| 1 | 0 | 1 | | | | | |
| 1 | 1 | 0 | | | | | |
| 1 | 1 | 1 | | | | | |

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| $a$ | $b$ | $c$ | $b+c$ | $a \cdot (b+c)$ | $a \cdot b$ | $a \cdot c$ | $a \cdot b + a \cdot c$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

$a \cdot (b+c)$ and $a \cdot b + a \cdot c$: same

**Claim:** $\bar{\bar{a}} = a$

**Pf:**

$$a \cdot b = b \cdot a$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
$$1 \cdot a = a$$
$$0 \cdot a = 0$$
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$
$$a \cdot a = a$$
$$a \cdot \bar{a} = 0$$
$$\overline{a \cdot b} = \bar{a} + \bar{b}$$

$$a + b = b + a$$
$$(a + b) + c = a + (b + c)$$
$$0 + a = a$$
$$1 + a = 1$$
$$a + (b \cdot c) = (a + b) \cdot (a + c)$$
$$a + a = a$$
$$a + \bar{a} = 1$$
$$\overline{(a + b)} = \bar{a} \cdot \bar{b}$$

**Claim:** $\overline{\overline{a}} = a$

**Pf:**

$$\overline{\overline{a}} = \overline{\overline{a \cdot a}}$$

$$= \overline{\overline{a} + \overline{a}}$$

$$= a \cdot a$$

$$= a$$

$$a \cdot b = b \cdot a$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$1 \cdot a = a$$

$$0 \cdot a = 0$$

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

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$$a + a = a$$

$$a + \overline{a} = 1$$

$$\overline{(a + b)} = \overline{a} \cdot \overline{b}$$

<u>Claim:</u> $\overline{0} = 1$

<u>Pf:</u>

$$a \cdot b = b \cdot a$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
$$1 \cdot a = a$$
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$$a + a = a$$
$$a + \bar{a} = 1$$
$$\overline{(a + b)} = \bar{a} \cdot \bar{b}$$

Claim: $\overline{0} = 1$

Pf:
$$\overline{0} = \overline{a \cdot \overline{a}} \quad \text{for some } a$$
$$= \overline{a} + a$$
$$= 1$$

$$a \cdot b = b \cdot a$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
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$$a + a = a$$
$$a + \overline{a} = 1$$
$$\overline{(a + b)} = \overline{a} \cdot \overline{b}$$

**Claim:** $a \cdot b + a \cdot \bar{b} = a$

**Pf:**

$$a \cdot b = b \cdot a$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
$$1 \cdot a = a$$
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$$a + \bar{a} = 1$$
$$\overline{(a + b)} = \bar{a} \cdot \bar{b}$$

Claim: $a \cdot b + a \cdot \bar{b} = a$

Pf: $a \cdot b + a \cdot \bar{b}$

$= a \cdot (b + \bar{b})$

$= a \cdot 1$

$= 1 \cdot a$

$= a$

$a \cdot b = b \cdot a$

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

$1 \cdot a = a$

$0 \cdot a = 0$

$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

$a \cdot a = a$

$a \cdot \bar{a} = 0$

$\overline{a \cdot b} = \bar{a} + \bar{b}$

$a + b = b + a$

$(a + b) + c = a + (b + c)$

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**Claim:** $a + a \cdot b = a$

**PF:**

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**Claim:** $a + a \cdot b = a$

**PF:**

$$
\begin{aligned}
a + a \cdot b &= a \cdot 1 + a \cdot b \\
&= a \cdot (1 + b) \\
&= a \cdot 1 \\
&= a
\end{aligned}
$$

$$a \cdot b = b \cdot a$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$
$$1 \cdot a = a$$
$$0 \cdot a = 0$$
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$
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$$a + a = a$$
$$a + \bar{a} = 1$$
$$\overline{(a + b)} = \bar{a} \cdot \bar{b}$$

- Let's think about rule $a \cdot b + a \cdot \bar{b} = a$
- This means we can rewrite

$$l_1 \cdot l_2 \cdot \cdots \cdot x_i \cdot \cdots l_{n-1} \cdot l_n$$
$$+$$
$$l_1 \cdot l_2 \cdot \cdots \cdot \bar{x}_i \cdot \cdots \cdot l_{n-1} \cdot l_n$$

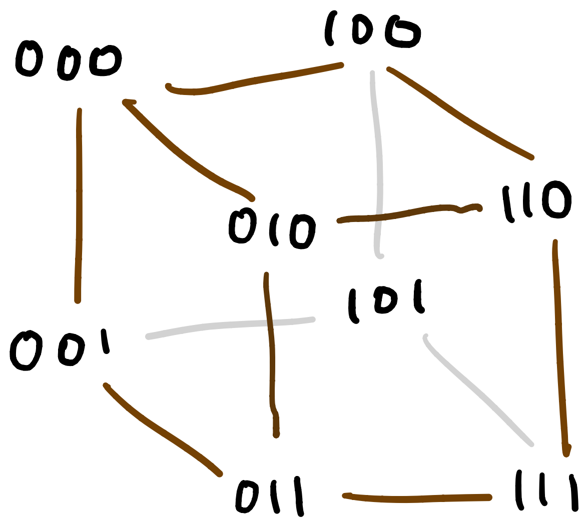
as $l_1 \cdot \cdots \cdot l_{i-1} \cdot l_{i+1} \cdot \cdots \cdot l_n$

smaller product "covers" * and ‡

truth table ...

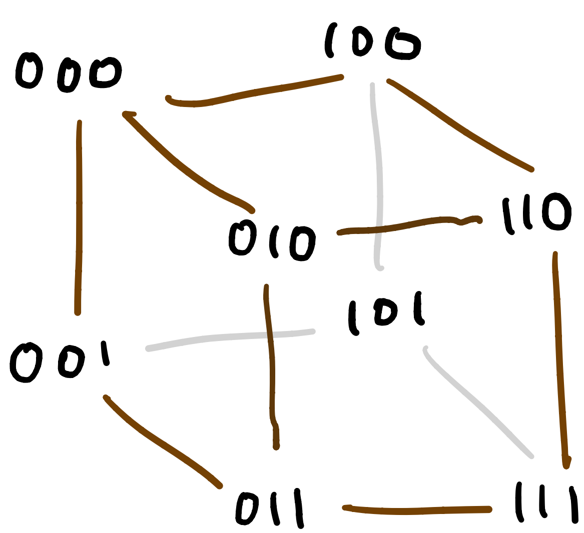
| | |
|---|---|
| ⋮ | ⋮ |
| $b_1 b_2 \cdots 0 \cdots b_{n-1} b_n$ | 1 * |
| ⋮ | ⋮ |
| $b_1 b_2 \cdots 1 \cdots b_{n-1} b_n$ | 1 ‡ |
| ⋮ | ⋮ |

This means that we should organize truth tables differently. Consider k-dimensional hypercube...




k = 3

This means that we should organize truth tables differently. Consider k-dimensional hypercube...




$k = 3$

truth table for 2:1 MUX

| s | $i_0$ | $i_1$ | o |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

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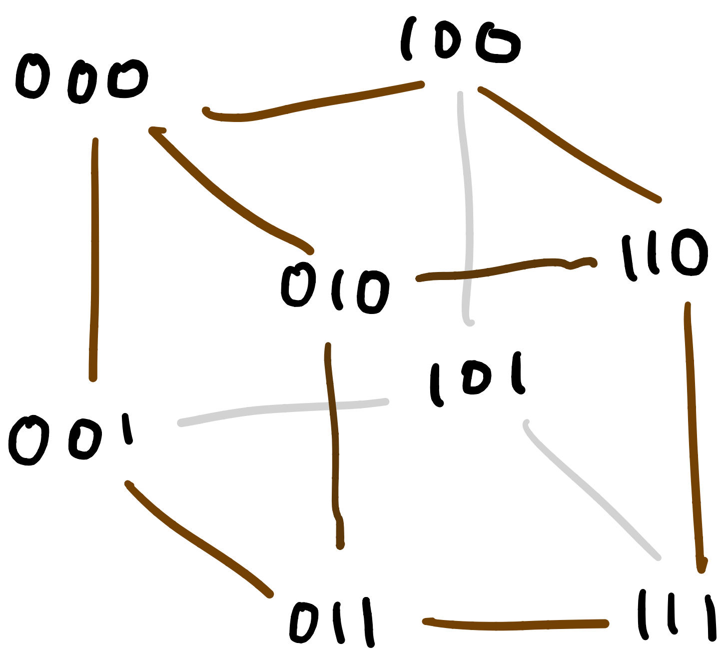


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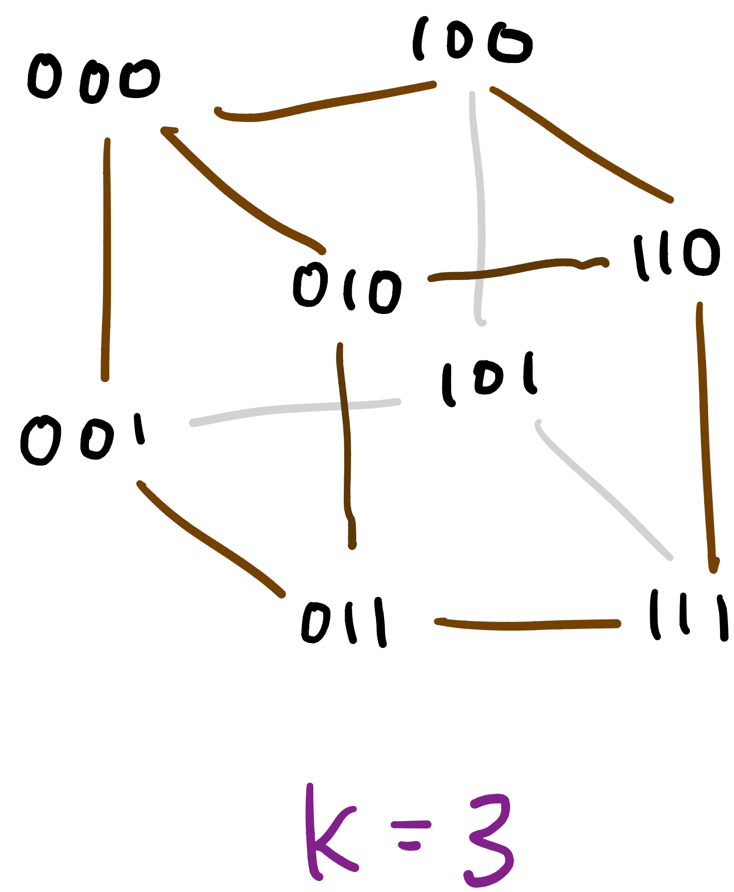


truth table for 2:1 MUX

| S | $i_0$ | $i_1$ | o |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
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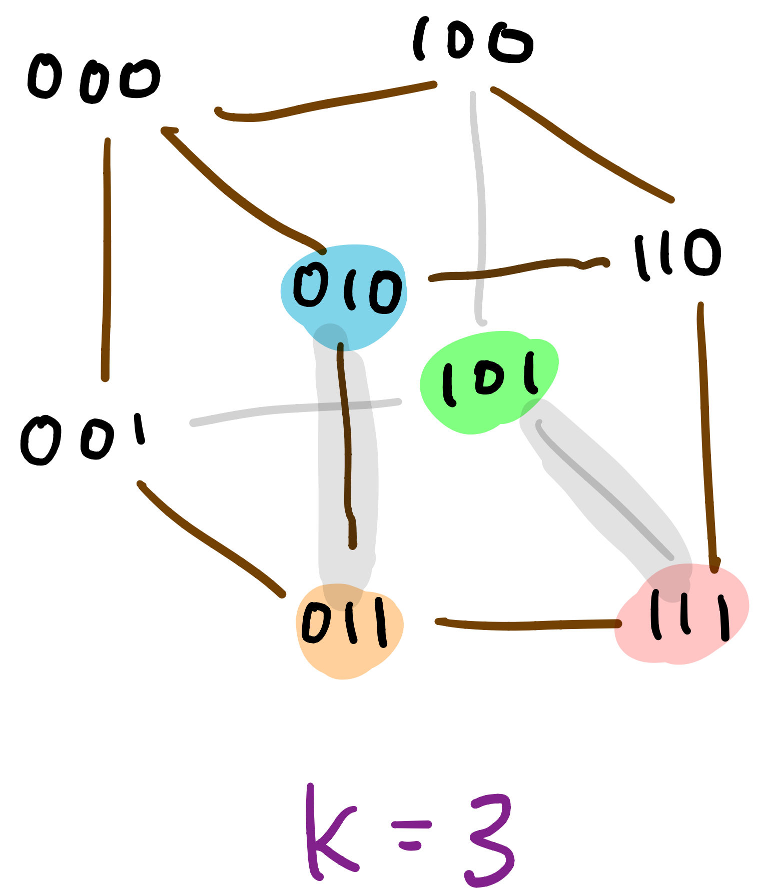


truth table for 2:1 MUX

| S | $i_0$ | $i_1$ | o |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
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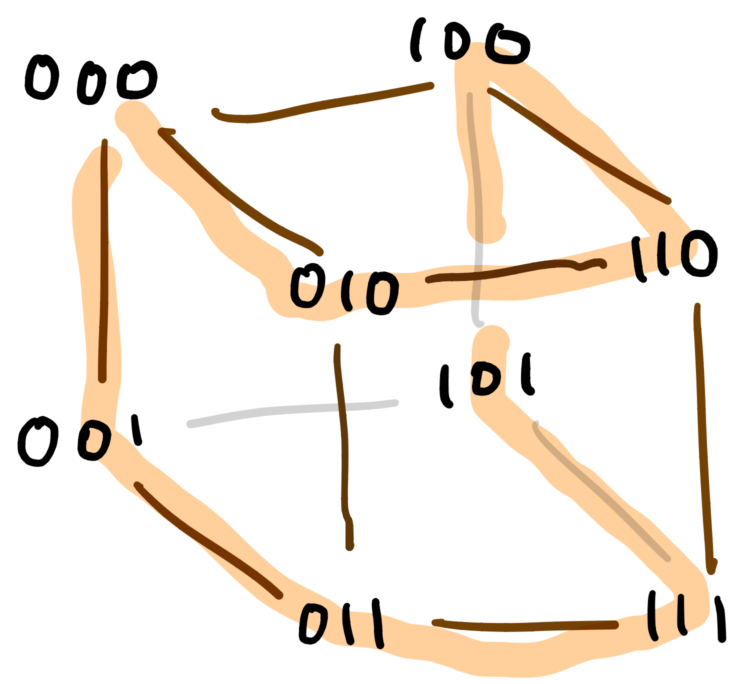
| 1 | 1 | 1 | 1 |

Defn: a Gray code is a cyclic sequence of all $2^k$ bit patterns of length k, where consecutive patterns differ by only one bit



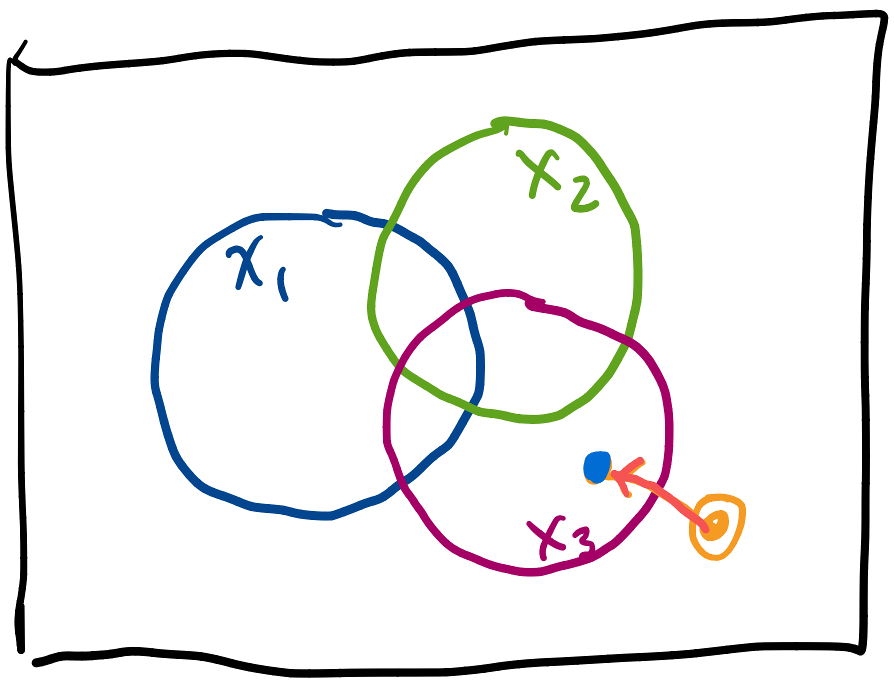

Note: a cyclic walk thru hypercube

E.g.

000, 001, 011, 111, 101, 100, 110, 010, 000, ...

**Defn**: a **Gray code** is a cyclic sequence of all $2^k$ bit patterns of length $k$, where consecutive patterns differ by only one bit

**Note**: It's also a walk thru a Venn diagram




**E.g.**

000, 001, 011, 111, 101, 100, 110, 010, 000, ...

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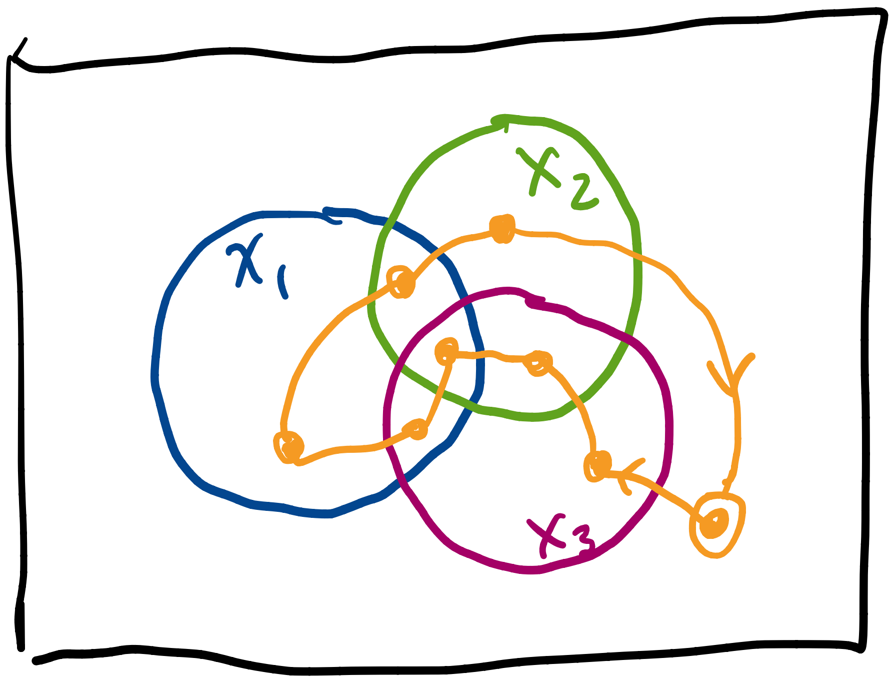


E.g.

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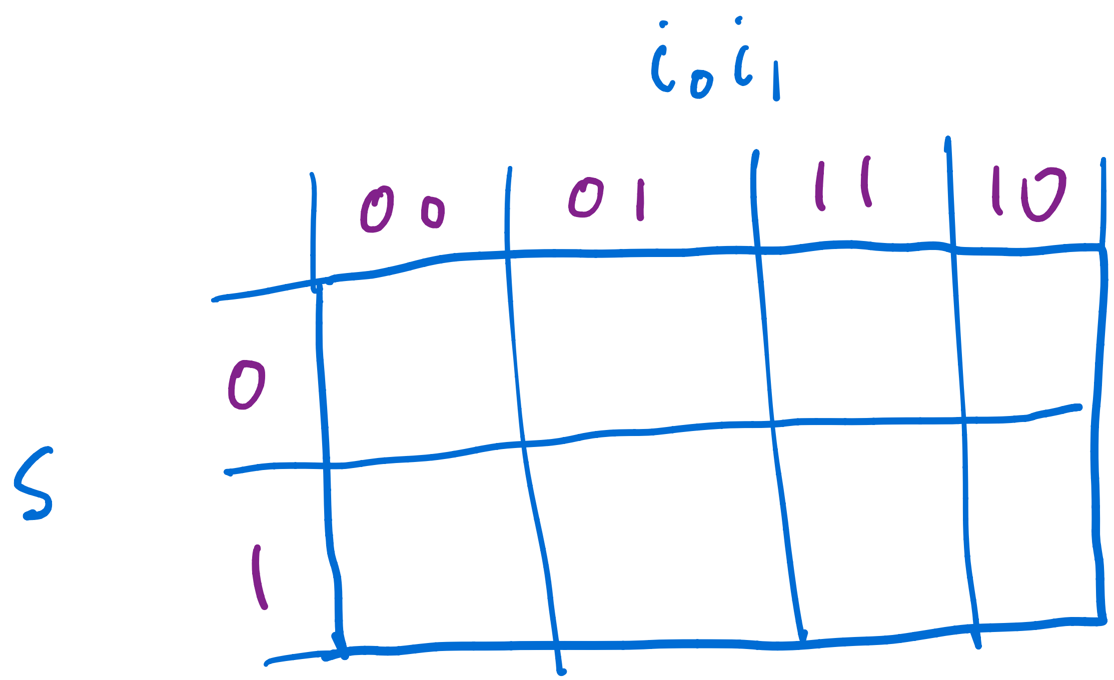
# Karnaugh Map

- a 2-dimensional unfolding of a hypercube
- a truth table with hypercube adjacencies



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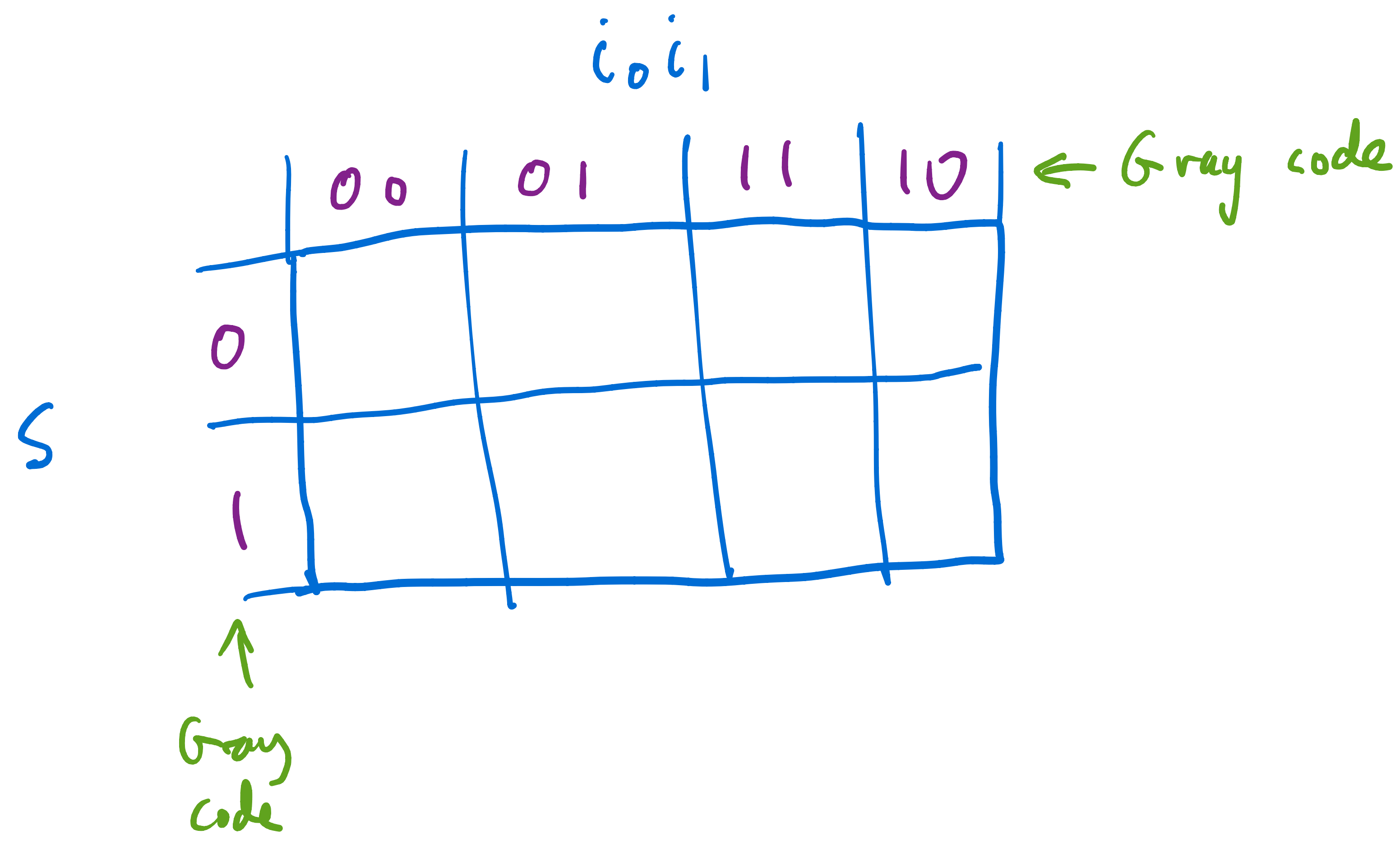

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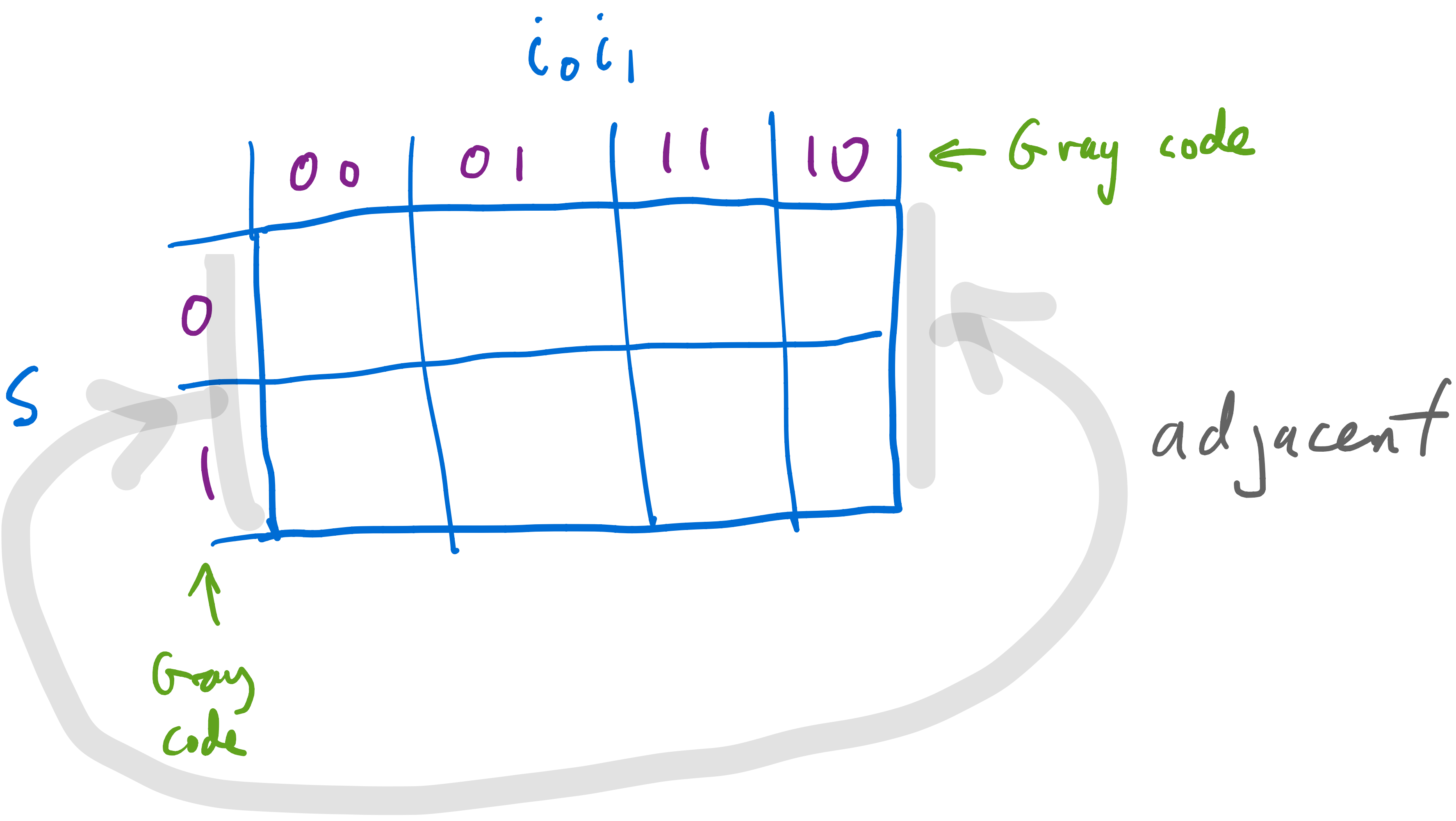

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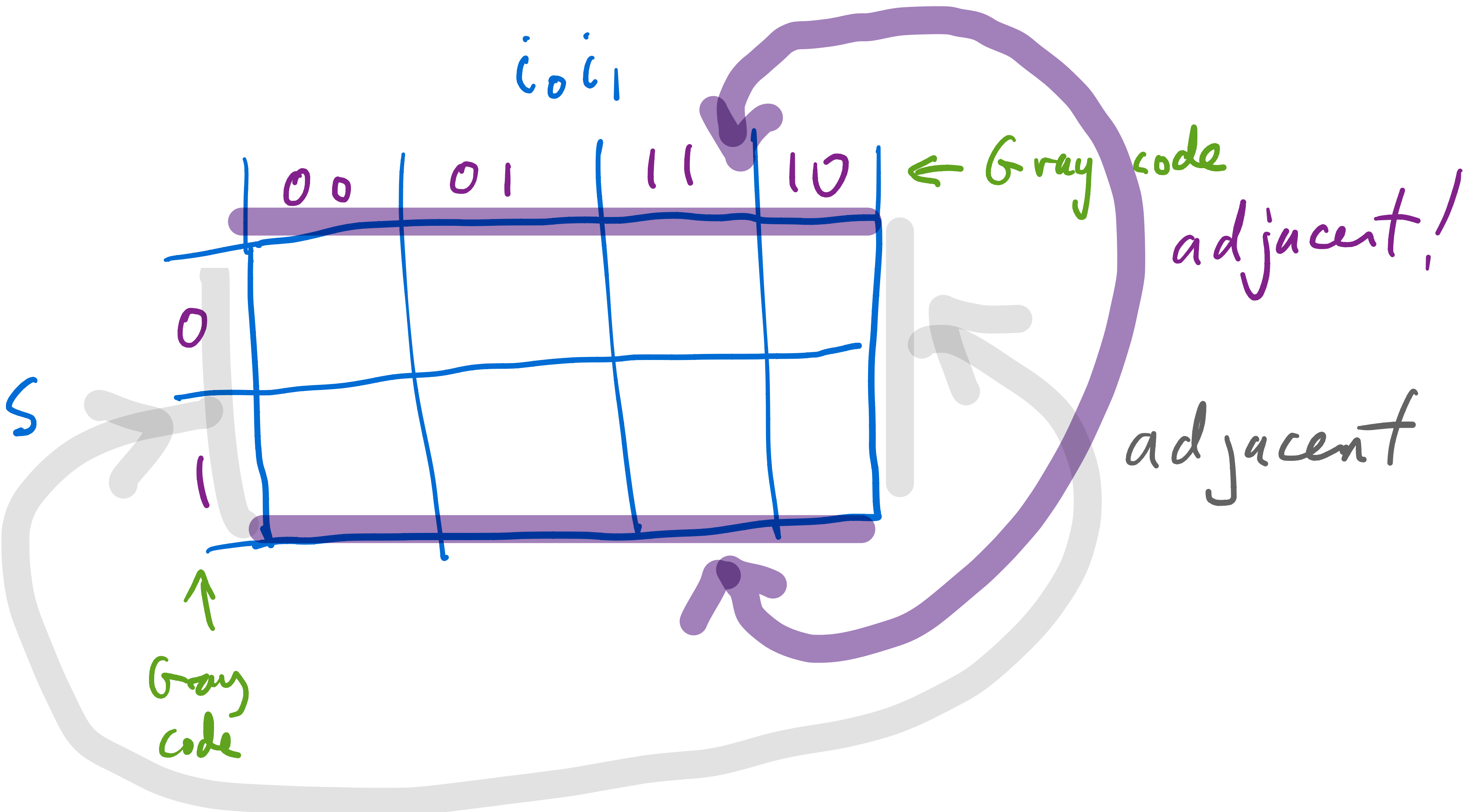

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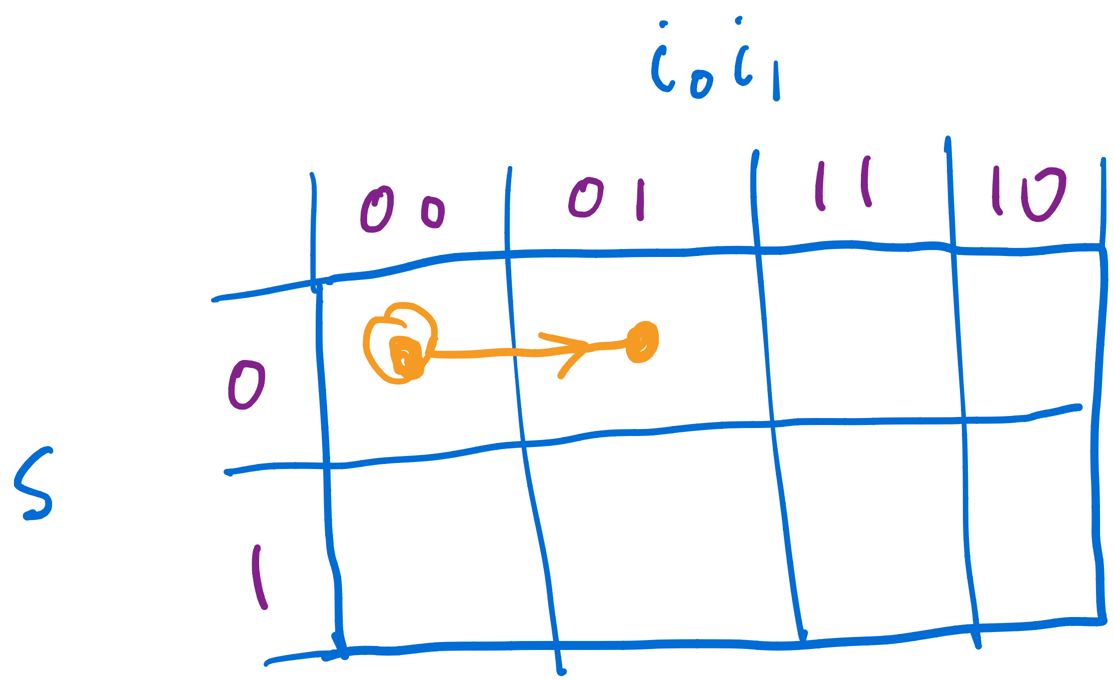


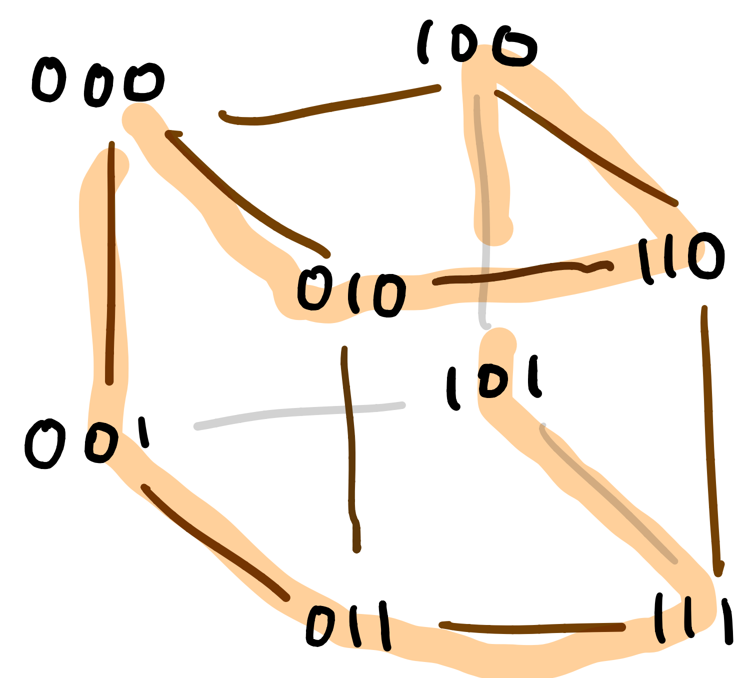


000, 001, 011, 111, 101,
100, 110, 010, 000, ...

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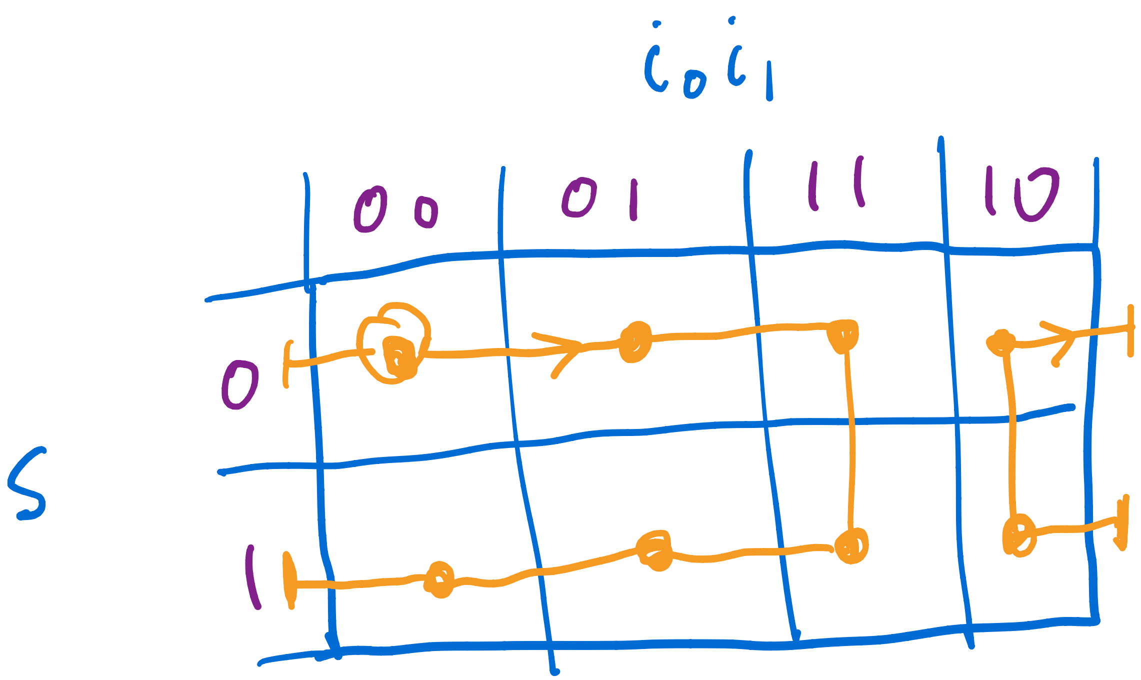


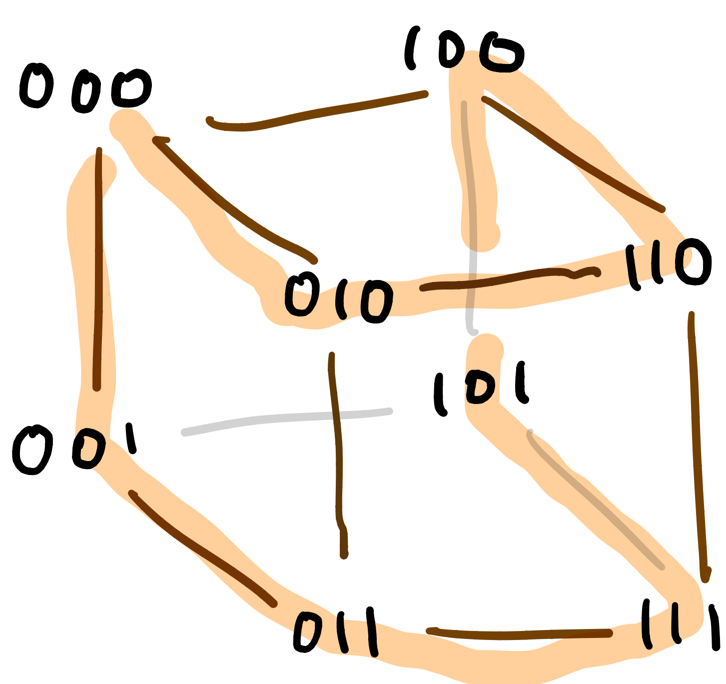


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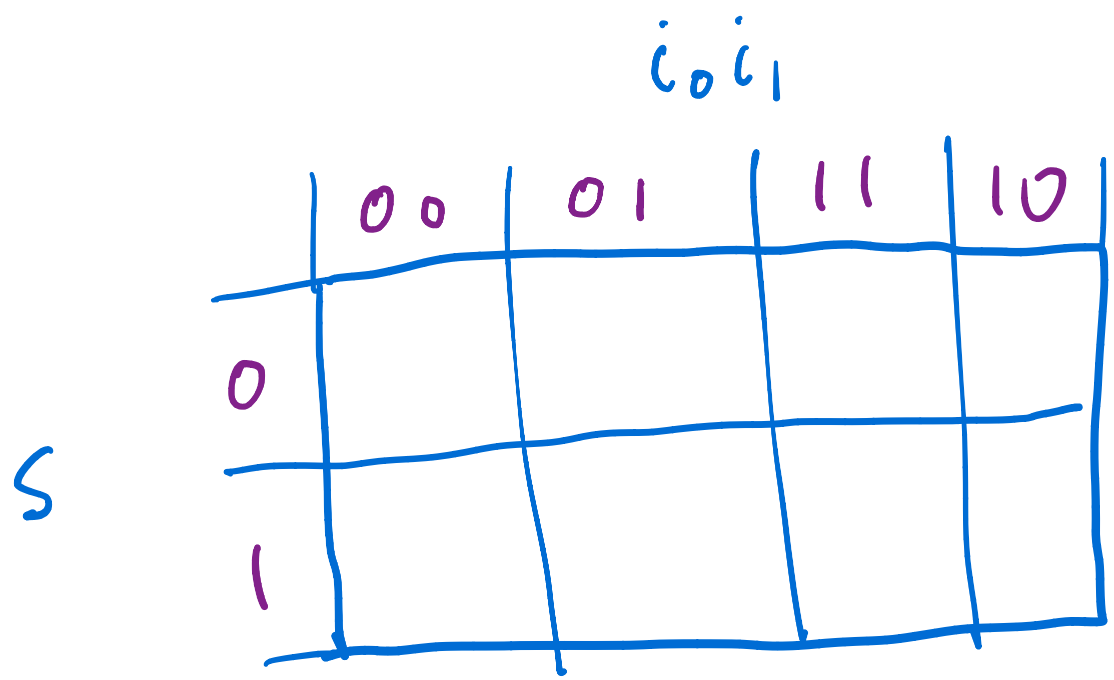
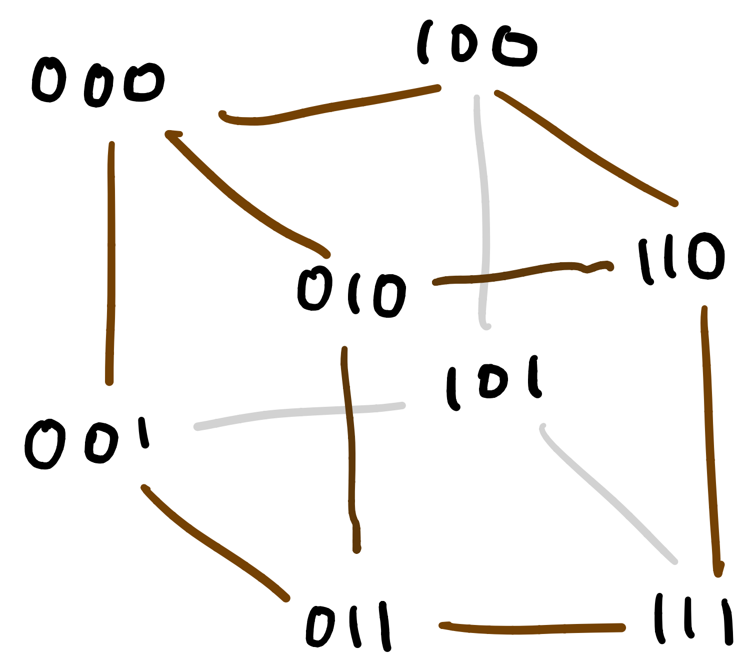


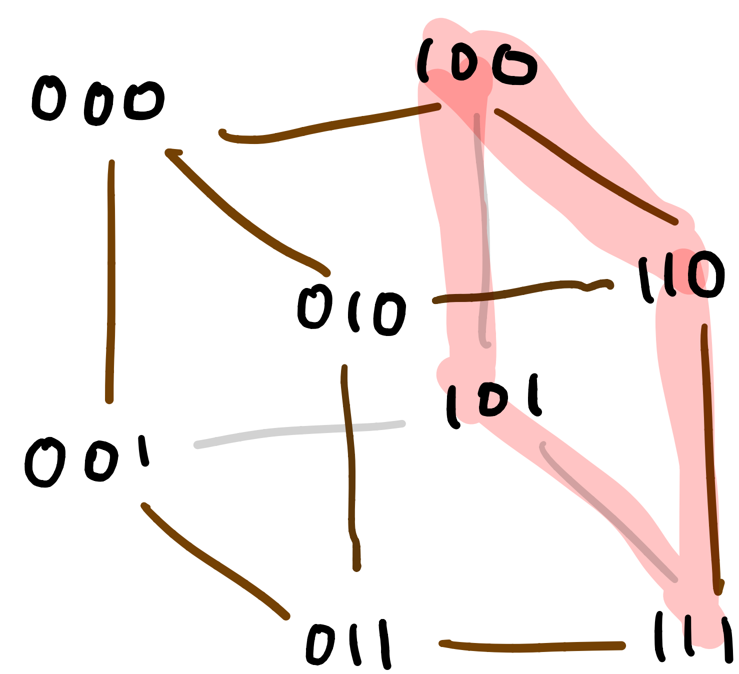




"subcubes" correspond to products...

# Karnaugh Map

- a 2-dimensional unfolding of a hypercube
- a truth table with hypercube adjacencies

| s \ $i_0 i_1$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | | | | |
| 1 | | | | |

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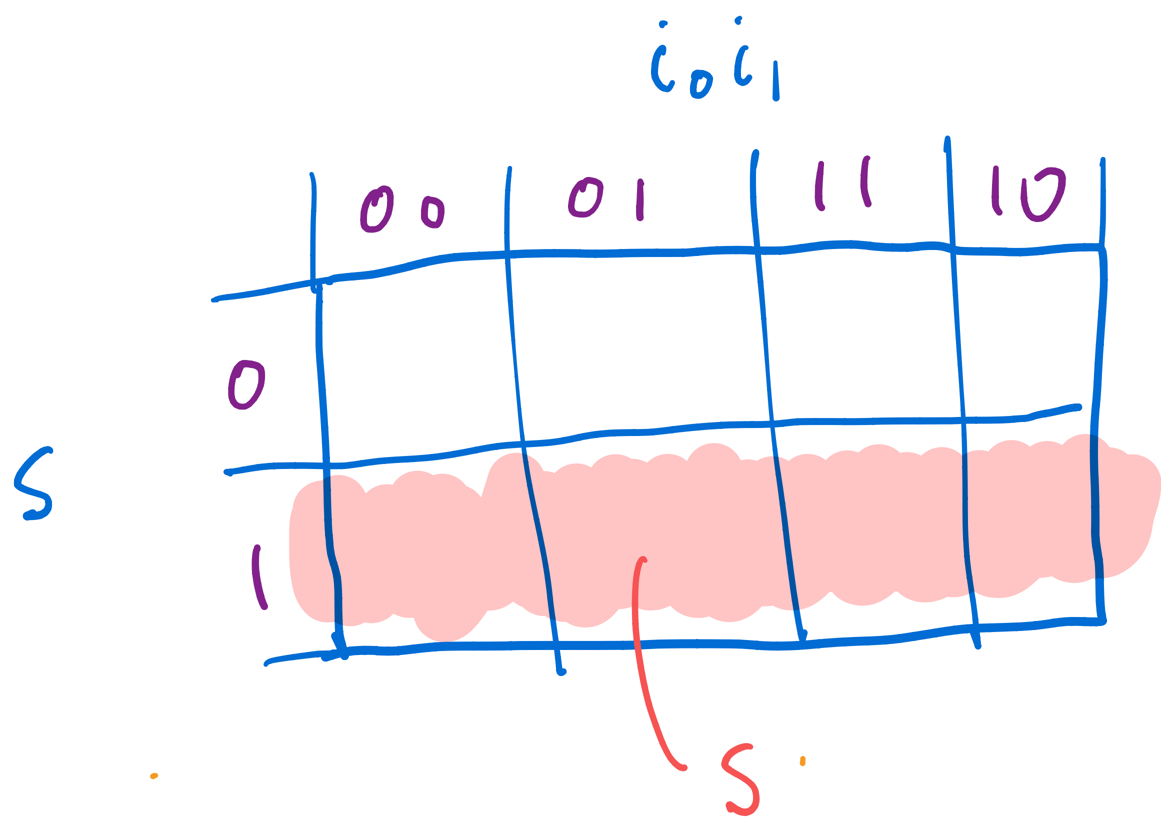


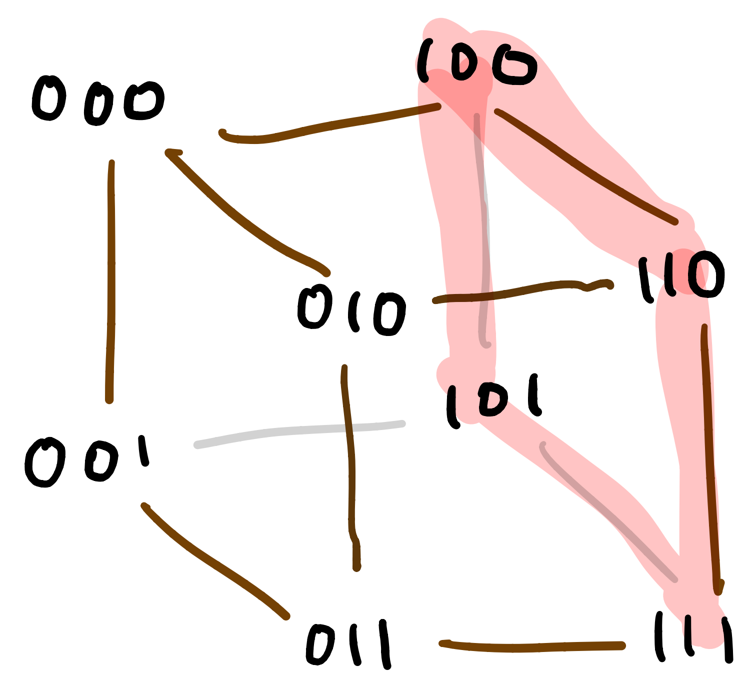


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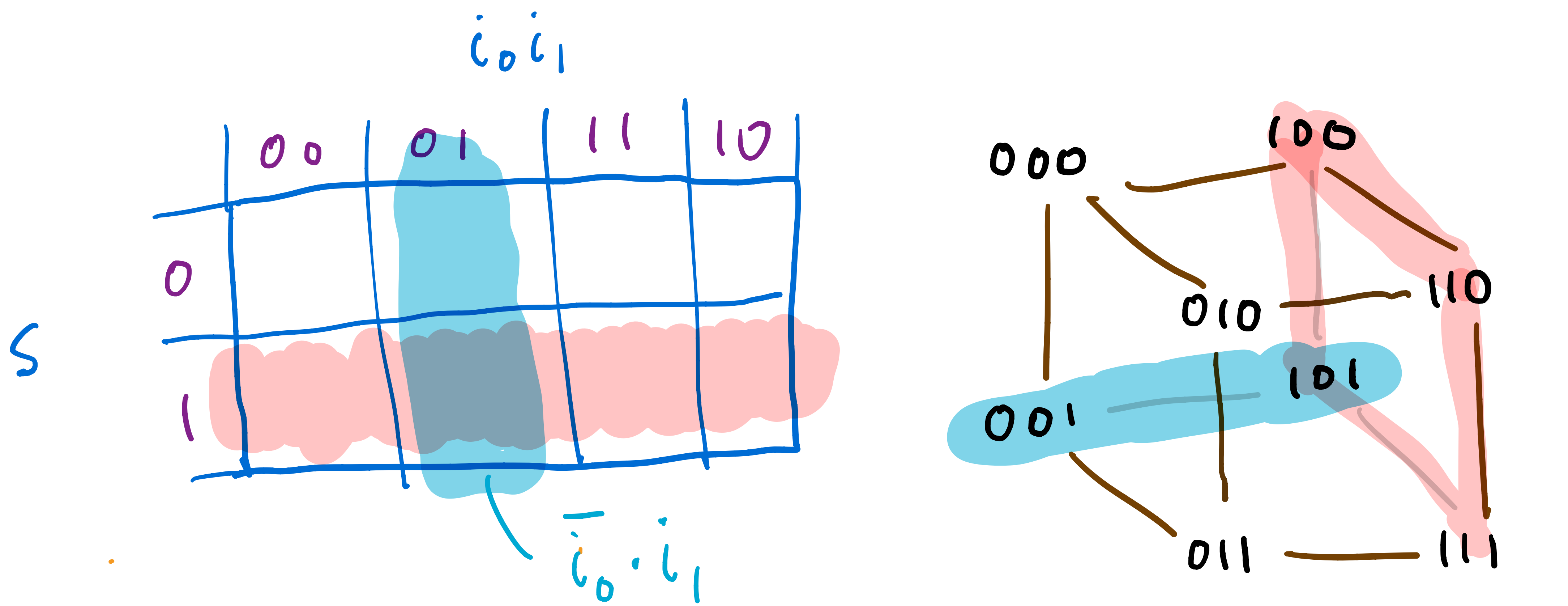


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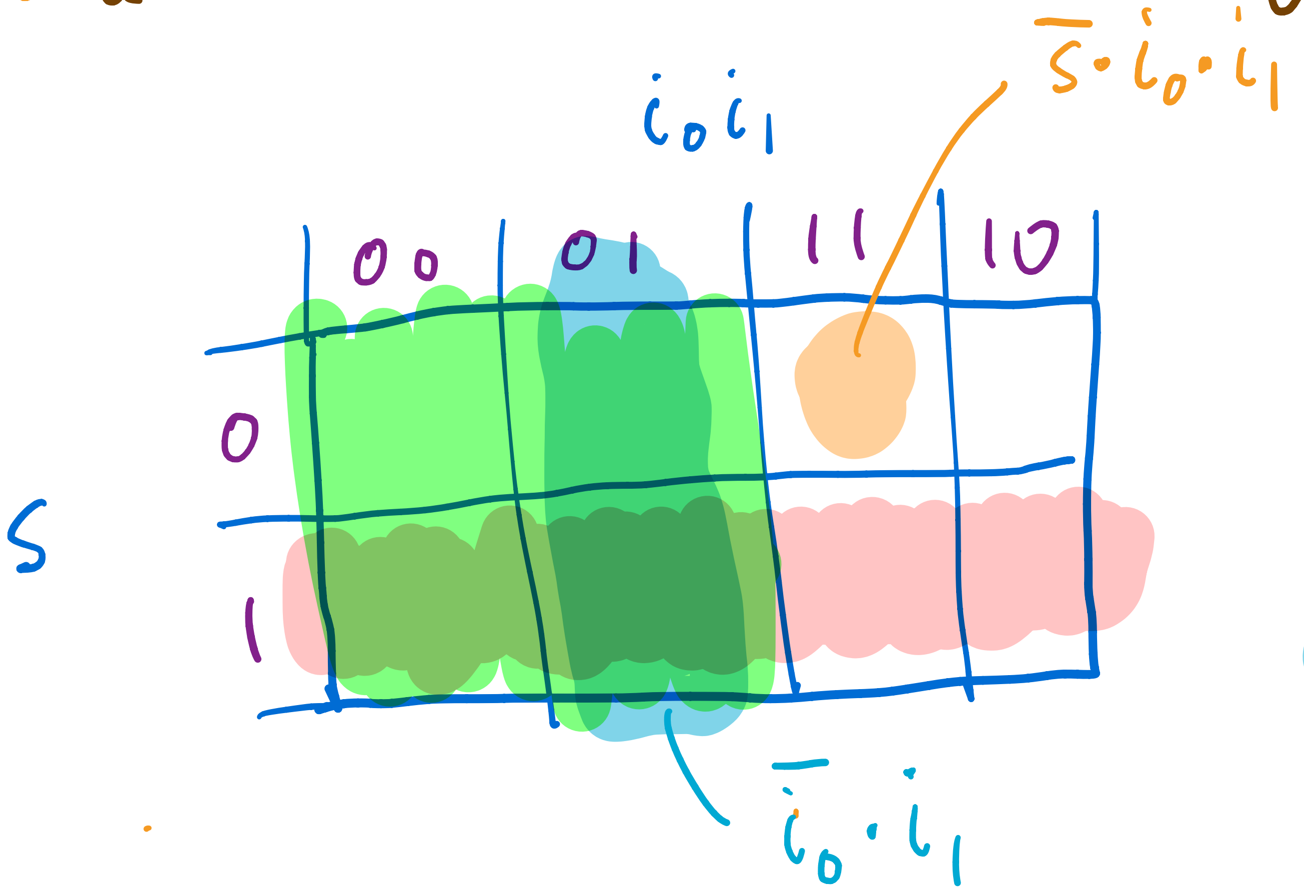


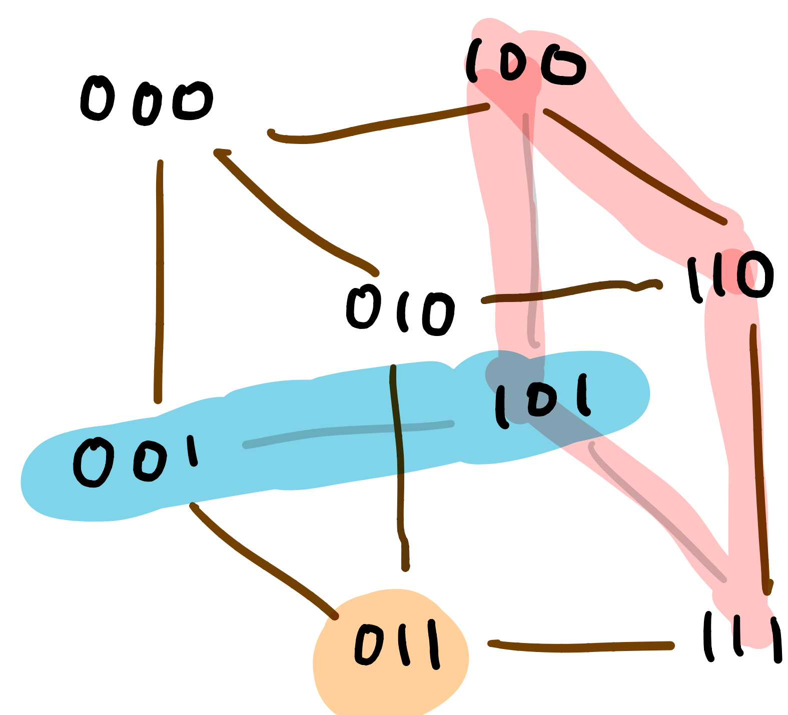

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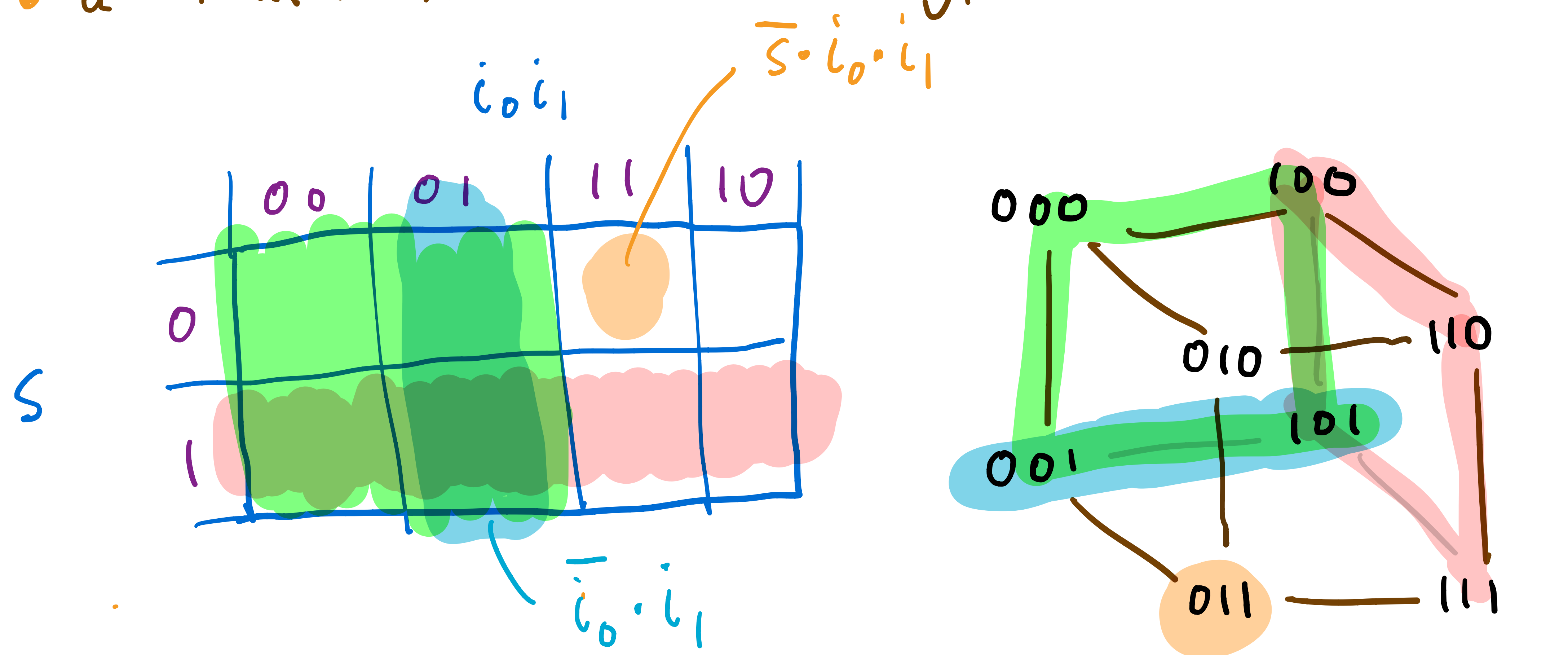
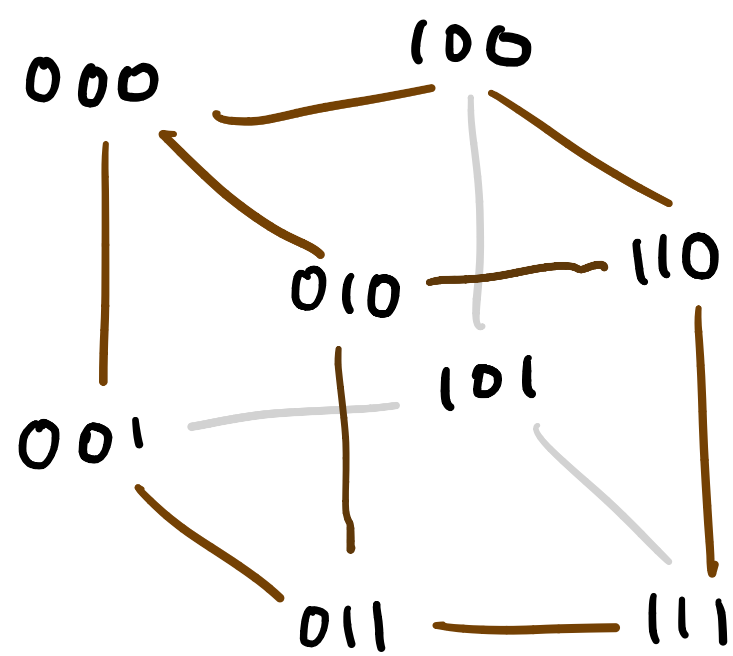


# Karnaugh Map

- a 2-dimensional unfolding of a hypercube
- a truth table with hypercube adjacencies

| s \ $i_0 i_1$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 |




Goal is to cover 1s w/ product rectangles...

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|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 0 |
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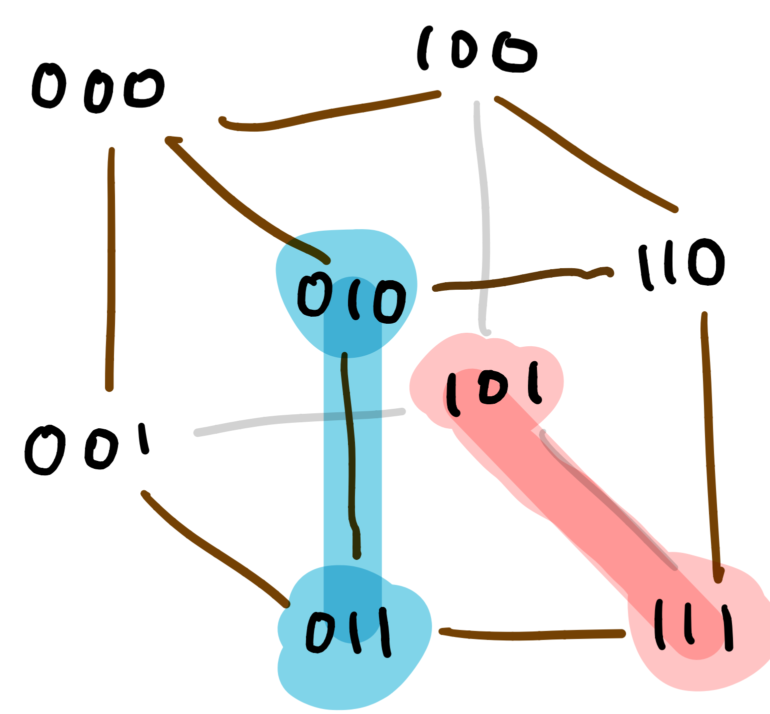



$$\mathrm{MUX2}(s, i_0, i_1) = s \cdot i_1 + \bar{s} \cdot i_0$$

$k=4$ example

| $x_1x_2$ \ $x_3x_4$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 0 | 0 |
| 01 | 0 | 1 | 1 | 0 |
| 11 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 0 | 1 |

K=4 example

| $x_1x_2$ \ $x_3x_4$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 0 | 0 |
| 01 | 0 | 1 | 1 | 0 |
| 11 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 0 | 1 |

K=4 example

| $x_1 x_2$ \ $x_3 x_4$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 0 | 0 |
| 01 | 0 | 1 | 1 | 0 |
| 11 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 0 | 1 |

$$x_1 \cdot \bar{x}_4 + \bar{x}_3 \cdot x_4 + x_2 x_4$$

Full adder:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| + 0 | + 1 | + 0 | + 1 | + 0 | + 1 | + 0 | + 1 |
| 0 0 | 0 1 | 0 1 | 1 0 | 0 1 | 1 0 | 1 0 | 1 1 |

$$\begin{array}{r} C_{in} \\ x \\ +\ y \\ \hline C_{out}\ z \end{array}$$

$$z := C_{in} \cdot \bar{x} \cdot \bar{y} + \bar{C}_{in} \cdot x \cdot \bar{y} + \bar{C}_{in} \cdot \bar{x} \cdot y + C_{in} \cdot x \cdot y$$

i.e. $z := \text{PARITY}(C_{in}, x, y)$

$$C_{out} := C_{in} \cdot x + x \cdot y + y \cdot C_{in}$$

```
  01011101
+ 11001110
----------
```

```
   1   1 1 1
   0 1 0 1 1 1 0 1
 + 1 1 0 0 1 1 1 0
 -----------------
 1 0 0 1 1 1 0 1 1
```

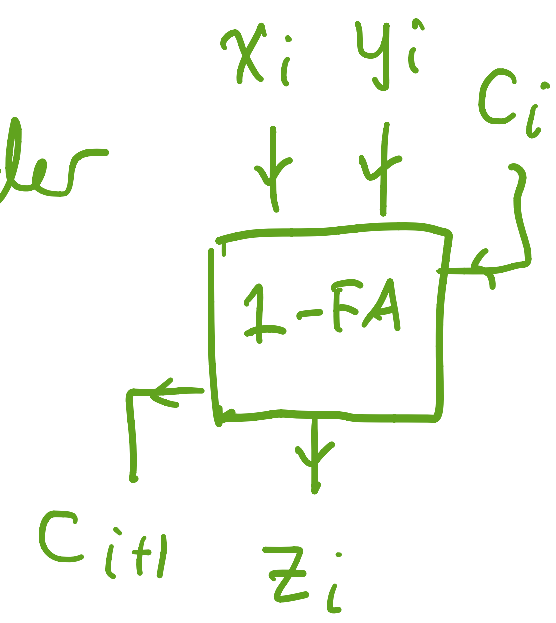
```
  1 1 0 1 1 1 0 0 0
    0 1 0 1 1 1 0 1
+   1 1 0 0 1 1 1 0
-------------------
  1 0 0 1 1 1 0 1 1
```

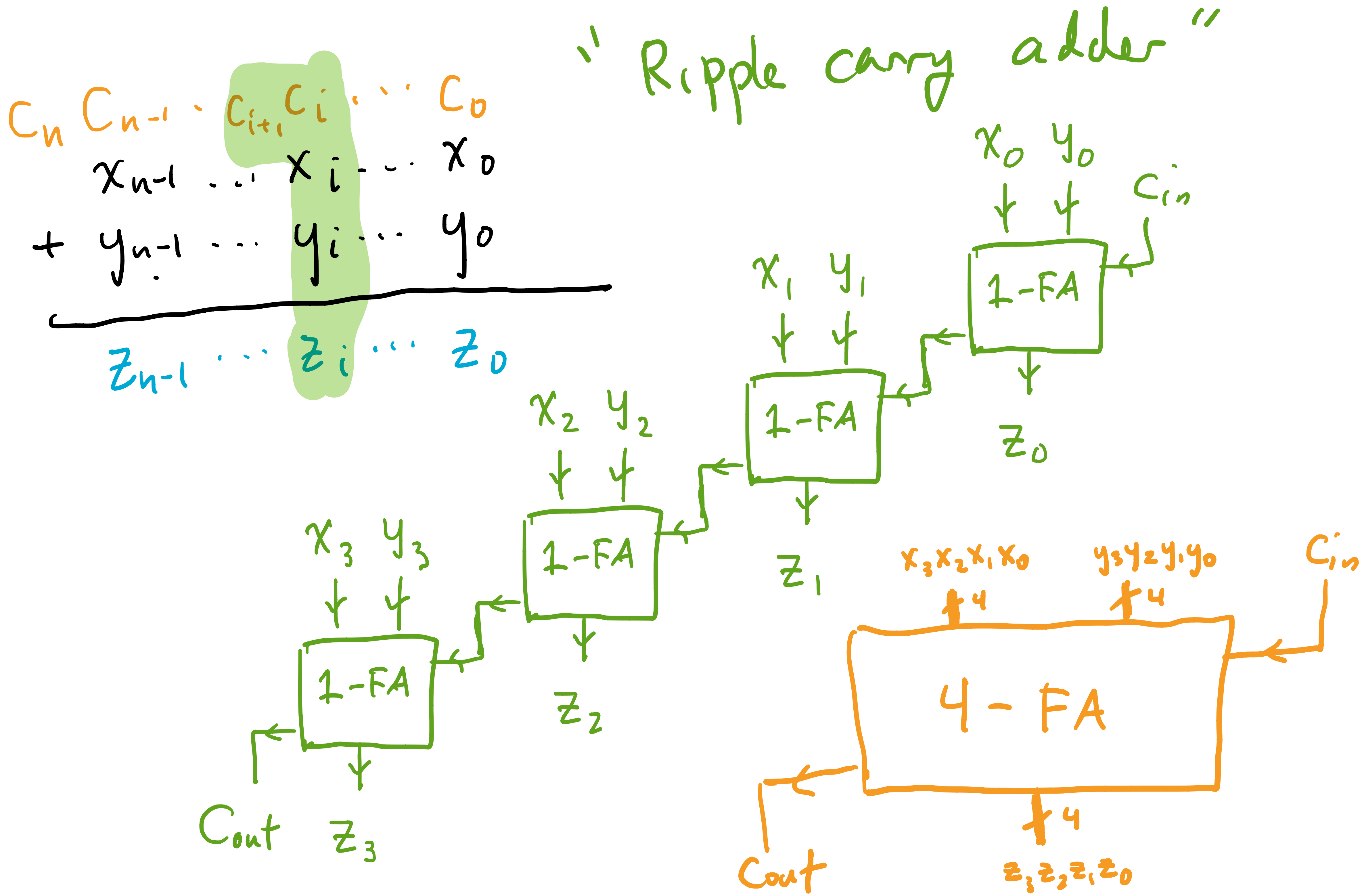
$$
\begin{array}{ccccccccc}
c_n & c_{n-1} & \cdots & c_{i+1} & c_i & \cdots & c_0 \\
 & x_{n-1} & \cdots & & x_i & \cdots & x_0 \\
+ & y_{n-1} & \cdots & & y_i & \cdots & y_0 \\
\hline
 & z_{n-1} & \cdots & & z_i & \cdots & z_0
\end{array}
$$

```
  1 1 0 1 1 1 0 0 0
    0 1 0 1 1 1 0 1
+   1 1 0 0 1 1 1 0
-------------------
  1 0 0 1 1 1 0 1 1
```

full adder

$$
\begin{array}{rcccccc}
c_n & c_{n-1} & \cdots & c_{i+1} & c_i & \cdots & c_0 \\
 & x_{n-1} & \cdots & & x_i & \cdots & x_0 \\
+ & y_{n-1} & \cdots & & y_i & \cdots & y_0 \\
\hline
 & z_{n-1} & \cdots & & z_i & \cdots & z_0
\end{array}
$$





"Ripple carry adder"
$C_n$ $C_{n-1}$ · $C_{i+1}$ $C_i$ · · · $C_0$
$x_{n-1}$ · · · $x_i$ · · · $x_0$
+ $y_{n-1}$ · · · $y_i$ · · · $y_0$
$z_{n-1}$ · · · $z_i$ · · · $z_0$
$x_0$ $y_0$
$C_{in}$
1-FA
$z_0$
$x_1$ $y_1$
1-FA
$z_1$
$x_2$ $y_2$
1-FA
$z_2$
$x_3$ $y_3$
1-FA
$C_{out}$ $z_3$
$x_3x_2x_1x_0$ 4
$y_3y_2y_1y_0$ 4
$C_{in}$
4-FA
$C_{out}$
4
$z_3z_2z_1z_0$

Q: How do we perform subtraction?

Related Q: How do we represent negative integers?

Q: How do we perform subtraction?

Related Q: How do we represent negative integers?

A: two's complement signed representation

$$x_{n-1}\, x_{n-2} \cdots x_2\, x_1\, x_0$$

sign bit

Recall: binary representation of "unsigned" integers

bit sequence is $x_{n-1} x_{n-2} \cdots x_2 x_1 x_0$

↑ most significant bit

coefficient of $2^{n-1}$

**Recall:** binary representation of "unsigned" integers

bit sequence is $x_{n-1} x_{n-2} \cdots x_2 x_1 x_0$

$x_{n-1}$: most significant bit, coefficient of $2^{n-1}$

$x_0$: least significant bit, coefficient of $2^0$

Recall: binary representation of "unsigned" integers

bit sequence is $x_{n-1} x_{n-2} \cdots x_i \cdots x_2 x_1 x_0$

most significant bit
coefficient of $2^{n-1}$

Coefficient of $2^i$

least significant bit
coefficient of $2^0$

Recall: binary representation of "unsigned" integers

bit sequence is $x_{n-1} x_{n-2} \ldots x_i \ldots x_2 x_1 x_0$

($x_i$: coefficient of $2^i$)

Represents number

$$v = x_{n-1} \cdot 2^{n-1} + x_{n-2} \cdot 2^{n-2} + \ldots + 2^i x_i + \ldots + 4x_2 + 2x_1 + 1x_0$$

E.g. 10110111 represents $128 + 32 + 16 + 4 + 2 + 1 = 183$

$$1 \cdot 128 + 0 \cdot 64 + 1 \cdot 32 + 1 \cdot 16 + 0 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 1 \cdot 1$$

Binary sequences of $n=3$

1 1 1
1 1 0
1 0 1
1 0 0
0 1 1
0 1 0
0 0 1
0 0 0

Binary sequences of $n=3$

| | represents... |
|---|---|
| 1 1 1 | 7 |
| 1 1 0 | 6 |
| 1 0 1 | 5 |
| 1 0 0 | 4 |
| 0 1 1 | 3 |
| 0 1 0 | 2 |
| 0 0 1 | 1 |
| 0 0 0 | 0 |

Binary sequences of n=3

| | represents... | |
|---|---|---|
| 1 1 1 | 7 | -1 |
| 1 1 0 | 6 | -2 |
| 1 0 1 | 5 | -3 |
| 1 0 0 | 4 | -4 |
| 0 1 1 | 3 | 3 |
| 0 1 0 | 2 | 2 |
| 0 0 1 | 1 | 1 |
| 0 0 0 | 0 | 0 |
| | unsigned interpretation | signed 2's complement interpretation |

!!!

Two's complement encoding...

| | |
|---|---|
| 1 1 1 | -1 |
| 1 1 0 | -2 |
| 1 0 1 | -3 |
| 1 0 0 | -4 |
| 0 1 1 | 3 |
| 0 1 0 | 2 |
| 0 0 1 | 1 |
| 0 0 0 | 0 |

# Two's complement encoding...

| | |
|---|---|
| 011 | 3 |
| 010 | 2 |
| 001 | 1 |
| 000 | 0 |
| 111 | -1 |
| 110 | -2 |
| 101 | -3 |
| 100 | -4 |

- Values range from -4 to 3
  i.e. $-2^2$ to $2^2-1$.
- In general, $-2^{k-1}$ to $2^{k-1}-1$.
- In C++ int is 32 bits
  Values are $-2^{31}$ to $2^{31}-1$.
- The long type is 64 bits.
  Values are $-2^{63}$ to $2^{63}-1$.

(Note: unsigned int is 0 to $2^{32}-1$)

Two's complement encoding...

| | |
|---|---|
| 011 | 3 |
| 010 | 2 |
| 001 | 1 |
| 000 | 0 |
| 111 | -1 |
| 110 | -2 |
| 101 | -3 |
| 100 | -4 |

↖ leftmost bit tells us whether # is negative

# Two's complement encoding... negating

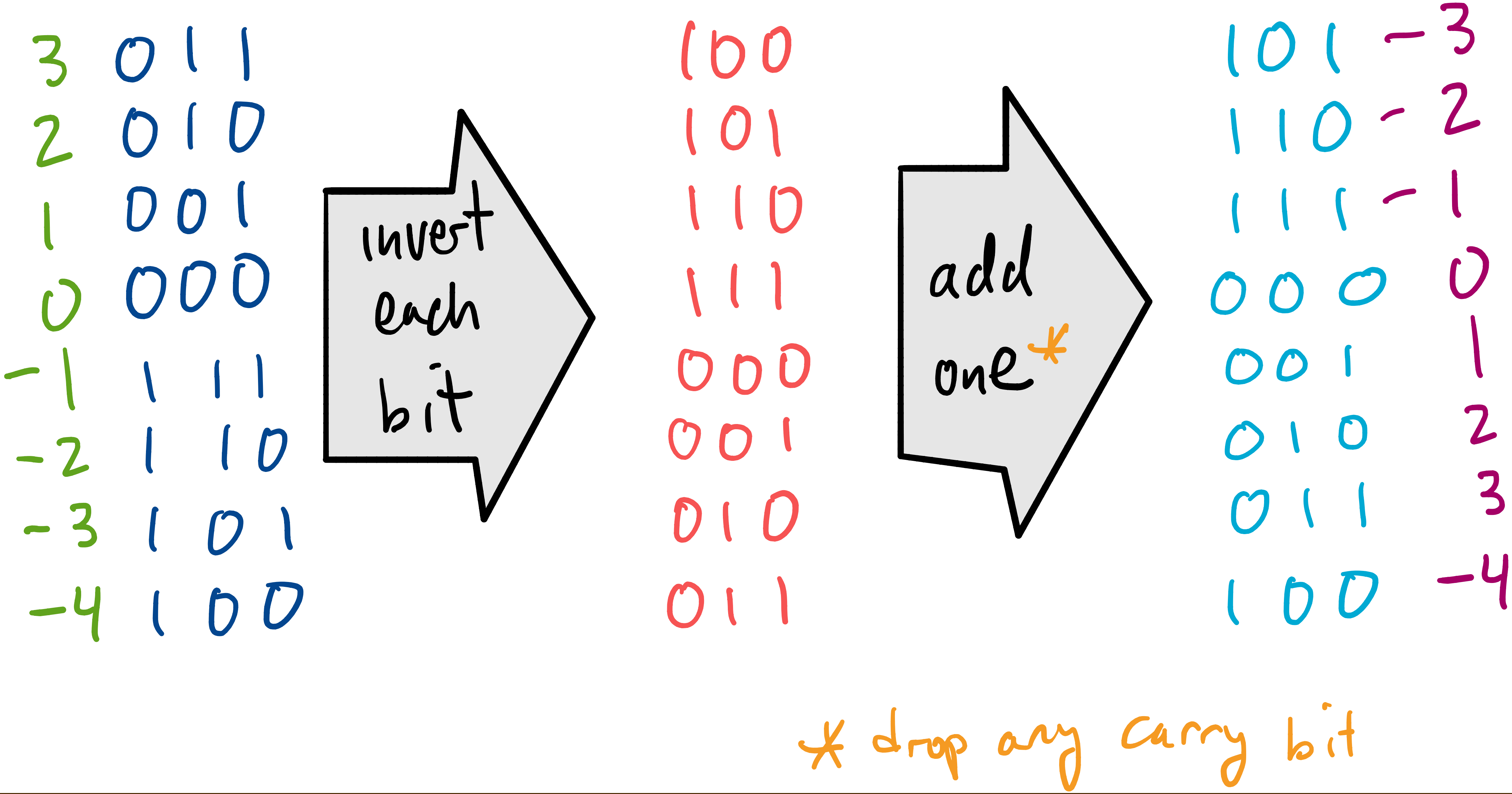
| | | | |
|---|---|---|---|
| 011 | 3 | -3 | |
| 010 | 2 | -2 | |
| 001 | 1 | -1 | |
| 000 | 0 | 0 | |
| 111 | -1 | 1 | |
| 110 | -2 | 2 | |
| 101 | -3 | 3 | |
| 100 | -4 | -4 | 100 |

# Two's complement encoding… negating

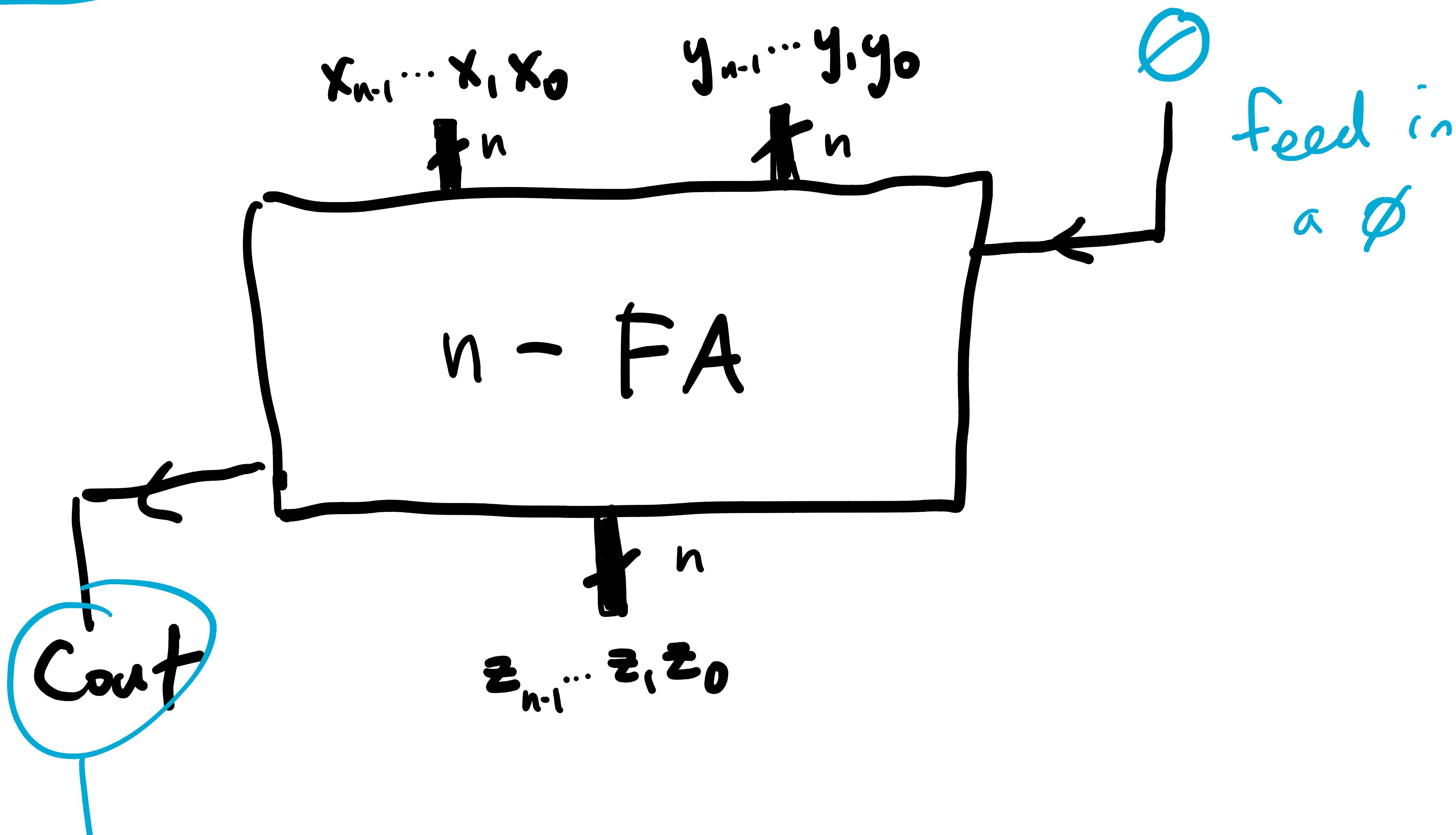
| | | | |
|---|---|---|---|
| 011 | 3 | -3 | 101 |
| 010 | 2 | -2 | 110 |
| 001 | 1 | -1 | 111 |
| 000 | 0 | 0 | 000 |
| 111 | -1 | 1 | 001 |
| 110 | -2 | 2 | 010 |
| 101 | -3 | 3 | 011 |
| 100 | -4 | -4 | 100 |

???

# Two's complement encoding... negating

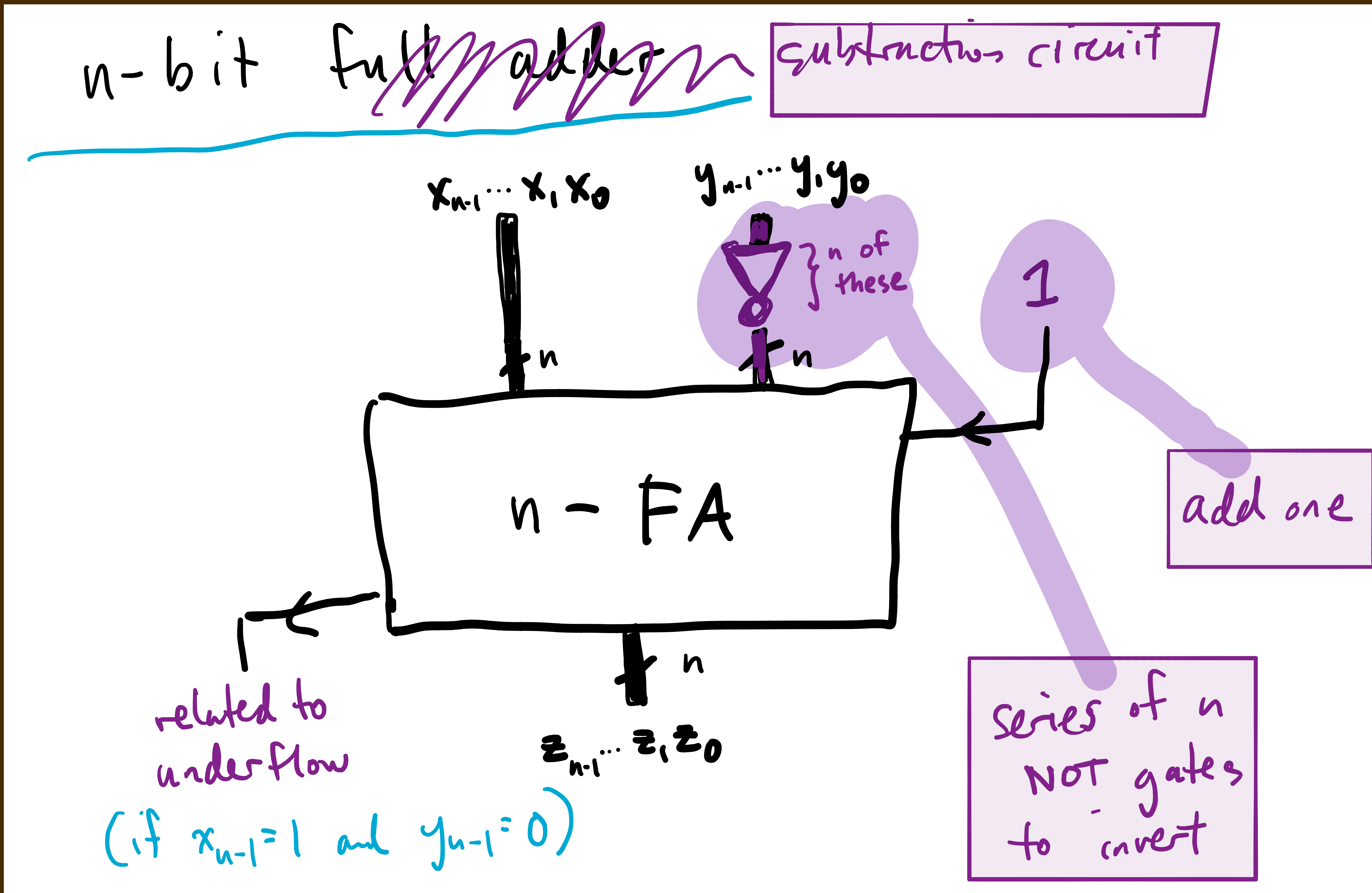


# n-bit full adder




overflow condition

with signed numbers if $x_{n-1} = y_{n-1} = 1$


n-bit full adder
subtraction circuit
x_{n-1} ... x_1 x_0
y_{n-1} ... y_1 y_0
n of these
n
n
1
n - FA
add one
n
z_{n-1} ... z_1 z_0
related to underflow
(if x_{n-1} = 1 and y_{n-1} = 0)
series of n NOT gates to invert

00 + 00 = 000

00 + 01 = 001

00 + 10 = 010

00 + 11 = 011

01 + 00 = 001

01 + 01 = 010

01 + 10 = 011

01 + 11 = 100

(1) + (−1) = (0)

10 + 00 = 010

10 + 01 = 011

10 + 10 = 100 ⊗

(−2) + (−2)

10 + 11 = 101 ⊗

(−2) + (−1)

11 + 00 = 011

11 + 01 = 100

(−1) + (+1) = (0)

11 + 10 = 101 ⊗

(−1) + (−2) = (1)

11 + 11 = 110

(−1) + (−1) = (−2)

00 + 00 = 000

00 + 01 = 001

00 + 10 = 010

00 + 11 = 011

01 + 00 = 001

01 + 01 = 010

01 + 10 = 011

01 + 11 = 100

1 −1 0

10 + 00 = 010

10 + 01 = 011

10 + 10 = 100 ⊗

−2 −2

10 + 11 = 101 ⊗

−2 −1

11 + 00 = 011

11 + 01 = 100

−1 +1 0

11 + 10 = 101 ⊗

−1 −2 1

11 + 11 = 110

−1 −1 −2